

Affine harmonic maps

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February 15, 2009

Summary: We introduce a class of maps from an affine flat into a Riemannian manifold that solve an elliptic system defined by the natural second order elliptic operator of the affine structure and the nonlinear Riemannian geometry of the target. These maps are called affine harmonic. We show an existence result for affine harmonic maps in a given homotopy class when the target has nonpositive sectional curvature and some global nontriviality condition is met. An example shows that such a condition is necessary.

The analytical part is made difficult by the absence of a variational structure underlying affine harmonic maps. We therefore need to combine estimation techniques from geometric analysis and PDE theory with global geometric considerations.

1 Introduction

A geometric structure usually induces a particular type of connection that preserves that structure. When we have a Riemannian geometry, we get the Levi-Civita connection as the unique torsion free connection that preserves the metric. For a complex structure, we get a canonical complex connection. For an affine structure – which is the type of structure interesting us in the present paper –, we obtain an affine flat connection. Thus, when we have different geometric structures on the same manifold, the induced connections then in general are also different. For instance, for a Hermitian metric on a complex manifold, its Levi-Civita connection will in general not coincide with the holomorphic connection. More precisely, the two coincide if and only if the manifold is Kähler. This compatibility between two structures then makes the theory of Kähler manifolds very rich. In fact, there is some analogy between Kähler and a particular class of affine structures first pointed out by Cheng and Yau [4]. Remarkably, these structures also arise from a completely different perspective, the one of information geometry, that is, a geometric view of statistical families, see [5, 2, 6, 11].

One of the motivations for the present work then is to develop appropriate tools from geometric analysis to investigate such structures. In Riemannian geometry, basic tools are geodesics and harmonic maps. Here, for instance, a geodesic can be defined either from a metric, as a curve that locally minimizes length, or from a connection, as an autoparallel curve. The first one is a variational characterization, the other is not. Likewise, harmonic maps are characterized by a variational principle involving the metric. Since harmonic maps are higher dimensional generalizations of geodesics, it is then natural to develop also the corresponding concepts in terms of a connection. This has been done by Jost-Yau [12] where the class of Hermitian harmonic maps is introduced. These maps are determined by the complex connection, and not by the Levi-Civita one. Therefore, they do not satisfy a variational principle, and their investigation becomes analytically much more difficult. Nevertheless, in [12], a complete analysis could be carried out. As for ordinary harmonic maps, it has to be required that the target manifold has nonpositive sectional curvature. Still, an example in [12] shows that in contrast to ordinary harmonic maps, a Hermitian harmonic map need not always exist in a given homotopy class, and a global nontriviality

condition needs to imposed to compensate for the lack of a variational structure.

In this paper, we introduce the corresponding concept of affine harmonic maps. They are determined in terms of an affine connection. Thus, they also in general lack a variational structure. In this paper, we succeed in extending the analysis of [12] to affine harmonic maps and to show a general existence theorem.

We hope that we can combine this existence results with Bochner type identities in order to derive new restrictions on the topology of affine flat manifolds.

2 Kähler affine and dually flat manifolds

We shall use the standard conventions for raising and lowering indices.

An affine manifold M possesses a covering by coordinate charts with affine coordinate changes. It then carries an affine flat connection, that is, one with vanishing curvature. This connection is complete if its geodesics can be defined on the real line. This is equivalent to the condition that the universal covering of M is an affine vector space which we identify with \mathbb{R}^n , with some abuse of notation. Note that compactness of M does not imply its completeness.

It has been an important research topic to derive restriction on affine manifolds under various restrictions on their fundamental group, see e.g. [13, 14, 7, 3].

Cheng and Yau [4] then introduced an important condition which they called Kähler affine: M carries a 2-tensor

$$\gamma_{\alpha\beta} dx^\alpha dx^\beta \quad (1)$$

which locally is of the form

$$\gamma_{\alpha\beta} = \frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} \quad (2)$$

for some convex function F , called a local potential (convexity here refers to local coordinates x and not to any metric.). Thus, γ is positive definite and symmetric, that is, defines a Riemannian metric on M . In general, of course, the Levi-Civita connection of γ will not be flat, that is, be different from the affine flat connection of M . The key point, however, is that the expression defining γ ,

$$\frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} dx^\alpha dx^\beta \quad (3)$$

is invariant under affine transformations.

We can, however, recover the flat connection from γ as follows: For $-1 \leq s \leq 1$, we define the s -connection through

$$\Gamma_{\alpha\beta\delta}^{(s)} = \Gamma_{\alpha\beta\delta}^{(0)} - \frac{s}{2} \partial_\alpha \partial_\beta \partial_\delta F \quad (4)$$

where $\Gamma_{\alpha\beta\delta}^{(0)}$ represents the Levi-Civita connection $\nabla^{(0)}$ for $\gamma_{\alpha\beta}$, i.e.,

$$\Gamma_{\alpha\beta\delta}^{(0)} = \langle \nabla_{\frac{\partial}{\partial x^\alpha}}^{(0)} \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\delta} \rangle. \quad (5)$$

Since by (2),

$$\Gamma_{\alpha\beta\delta}^{(0)} = \frac{1}{2} \partial_\alpha \partial_\beta \partial_\delta F, \quad (6)$$

we have

$$\Gamma_{\alpha\beta\delta}^{(s)} = \frac{1}{2} (1-s) \partial_\alpha \partial_\beta \partial_\delta F, \quad (7)$$

and since this is symmetric in α and β , $\nabla^{(s)}$ is torsion free. Since $\Gamma_{\alpha\beta\delta}^{(s)} + \Gamma_{\alpha\beta\delta}^{(-s)} = 2\Gamma_{\alpha\beta\delta}^{(0)}$, $\nabla^{(s)}$ and $\nabla^{(-s)}$ are dual to each other, in the sense that

$$Z\langle V, W \rangle = \langle \nabla_Z^{(s)} V, W \rangle + \langle V, \nabla_Z^{(-s)} W \rangle \quad (8)$$

for all vector fields V, W, Z where $\langle \cdot, \cdot \rangle$ stands for the metric g .

In particular, $\Gamma_{\alpha\beta\delta}^{(1)} = 0$, and so $\nabla^{(1)}$ defines a flat structure, and the coordinates x are affine coordinates for $\nabla^{(1)}$.

The connection dual to $\nabla^{(1)}$ then is $\nabla^{(-1)}$ with Christoffel symbols

$$\Gamma_{\alpha\beta\delta}^{(-1)} = \partial_\alpha \partial_\beta \partial_\delta F$$

with respect to the x -coordinates. We can then obtain dually affine coordinates ξ by

$$\xi_\beta = \partial_\beta F, \quad (9)$$

and so also

$$\gamma_{\alpha\beta} = \partial_\alpha \xi_\beta. \quad (10)$$

The corresponding local potential is obtained by a Legendre transformation

$$\Phi(\xi) = \max_x (x^\alpha \xi_\alpha - F(x)), \quad F(x) + \Phi(\xi) - x \cdot \xi = 0, \quad (11)$$

and

$$x^\beta = \partial^\beta \Phi(\xi), \quad \gamma^{\alpha\beta} = \frac{\partial x^\beta}{\partial \xi_\alpha} = \partial^\alpha \partial^\beta \Phi(\xi). \quad (12)$$

Thus, a Kähler affine structure yields a dually flat structure, i.e., a Riemannian metric γ together with two flat connections ∇ and ∇^* that are dual with respect to γ . Such dually flat structures have been introduced and investigated by Chensov [5] and Amari (see [2, 6]) as the basis of information geometry. Conversely, given such a dually flat structures, one finds local potential functions, that is, obtains a Kähler affine structure, see e.g. the exposition in [11]. Thus, the two types of structure are equivalent. Here, we work with the notion of Kähler affine structure of Cheng-Yau because it is geometrically simpler and more transparent.

Throughout this paper, we shall use standard summation conventions. c will denote a constant in estimates, without implying that c always has the same value. Being a constant here means that it depends only on the underlying geometries as well as possibly on the initial values, but not on the solutions of the differential equations under consideration.

3 Affine harmonic maps

Kähler affine structure (2) allows us to define a differential operator,

$$L := \gamma^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta}, \quad (13)$$

that is affinely invariant. A function $f : M \rightarrow \mathbb{R}$ that satisfies

$$Lf = 0 \quad (14)$$

is called *affine harmonic*. More generally, when N is a Riemannian manifold with metric g_{ij} and Christoffel symbols Γ_{jk}^i , we call a map $f : M \rightarrow N$ *affine harmonic* if it satisfies

$$\gamma^{\alpha\beta} \left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} + \Gamma_{jk}^i \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \right) = 0 \quad (15)$$

in local coordinates on N . More invariantly, we can write (15) as

$$\gamma^{\alpha\beta} D_\alpha D_\beta f = 0 \quad (16)$$

where D is the connection on $T^*M \otimes f^{-1}TN$ induced by the flat connection on M and the Levi-Civita connection on N .

We have the following general existence result for affine harmonic maps.

Theorem 3.1 *Let M be a compact Kähler affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \rightarrow N$ be continuous, and suppose g is not homotopic to a map $g_0 : M \rightarrow N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$.*

Then g is homotopic to an affine harmonic map $f : M \rightarrow N$.

Using the argument first introduced by Al'ber [1], one can also show that the affine harmonic map is unique in its homotopy class under the conditions of our theorem.

After stating some corollaries and discussing an example, we shall obtain this result in the next section by the method of [12].

Corollary 3.2 *Let M be a compact Kähler affine manifold, N a compact Riemannian manifold of negative sectional curvature. Let $g : M \rightarrow N$ be continuous, and suppose g is not homotopic to a map onto a closed geodesic of N . Then g is homotopic to an affine harmonic map.*

Corollary 3.3 *Let M be a compact Kähler affine manifold, N a compact Riemannian manifold of nonpositive sectional curvature. Let $g : M \rightarrow N$ be smooth and satisfy $e(g^*TN) \neq 0$, where e is the Euler class. Then g is homotopic to an affine harmonic map.*

The two corollaries follow from the theorem because their assumptions imply that g cannot be homotopic to a map $g_0 : M \rightarrow N$ for which there is a nontrivial parallel section of $g_0^{-1}TN$. In fact, for the first corollary, we observe that if the tangent space of $g_0(M)$ possesses a parallel section then $g_0(M)$ itself has to be a flat subspace of the nonpositively curved space N . Since N here is assumed to have negative curvature, the only such subspaces are one-dimensional, and they are homotopic to closed geodesics. For the second corollary, we observe that a vector bundle with a parallel section has vanishing Euler class.

(15) is a semilinear system of elliptic partial differential equations. It is in general not in divergence form, and therefore, variational methods are not available for its investigation. The method of [12] which we shall use for these existence theorems consists in studying the associated parabolic equation,

$$\frac{\partial f^i}{\partial t} = \gamma^{\alpha\beta} \left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} + \Gamma_{jk}^i \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \right) \quad (17)$$

for $f : M \times [0, \infty) \rightarrow N$ with initial values $f(x, 0) = g(x)$. A solution is shown to exist for all times $0 \leq t < \infty$ under the assumption that N has nonpositive sectional curvature and to converge to a solution of (15) for $t \rightarrow \infty$ under the geometric assumptions of the theorem or the corollaries. In order to see the relevance of these assumptions, let us consider the following example:

On \mathbb{R}^2 , consider the affine transformations

$$(x, y) \rightarrow (x + ny + m + \frac{1}{2}n^2, y + n) \quad (18)$$

for $m, n \in \mathbb{Z}$. The quotient of \mathbb{R}^2 by this action of \mathbb{Z}^2 then is a compact affine manifold M , see e.g. [7].

$$\tilde{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^1, (x, y) \mapsto x - \frac{1}{2}y^2 \quad (19)$$

then is a map which equivariant w.r.t. the homomorphism $(m, n) \rightarrow m$ (i.e., $\tilde{g}(x + ny + m + \frac{1}{2}n^2, y + n) = \tilde{g}(x, y) + m$ and therefore induces a map

$$g : M \rightarrow S^1 \quad (20)$$

where $S^1 = \mathbb{R}^1/\mathbb{Z}$. We consider the heat flow on \mathbb{R}^2 ,

$$\frac{\partial \phi}{\partial t} = \Delta \phi \quad (21)$$

with initial values $\phi(x, y, 0) = \tilde{g}(x, y)$ where Δ is the standard Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The solution of (21) is given by

$$\phi(x, y, t) = x - \frac{1}{2}y^2 - t \quad (22)$$

and therefore, it stays equivariant for all $t > 0$. For $t \rightarrow \infty$, it disappears at infinity and does not converge to a harmonic function.

This is not precisely the situation considered here because the Laplace operator Δ is not invariant under the action of \mathbb{Z}^2 on \mathbb{R}^2 , but since the solution ϕ nevertheless stays equivariant, this does not matter. (Actually, an invariant metric is given by

$$\gamma_{\alpha\beta}(x, y) = \begin{bmatrix} 1 & -y \\ -y & y^2 + 1 \end{bmatrix} \quad (23)$$

which is not Kähler affine.)

4 Proof of the main theorem

We shall abbreviate (17) as

$$\frac{\partial f}{\partial t} = \sigma(f). \quad (24)$$

Since this is a system of parabolic differential equations, the existence of a solution on a short time interval $[0, \tau)$ and, more generally, the openness of the existence interval follow from general results about parabolic equations. The first difficult step of the proof will now consist in showing the closedness of the existence interval. For that step, we shall need the nonpositive sectional curvature of the target. The second step will then be to show that the solution of (24) converges to an affine harmonic map as $t \rightarrow \infty$. For that, we need to show in particular that $f_t \rightarrow 0$ as $t \rightarrow \infty$. For that step, we shall need to use the homotopic nontriviality condition in addition to the nonpositive sectional curvature.

We now carry out the first step. It will be divided into several substeps.

1. Let $f(x, t, s)$ be a family of solutions of (24) depending on a parameter s . We then compute, using (24) to convert third derivatives into curvature terms by the standard commutation formula for covariant derivatives

$$\left(\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} - \frac{\partial}{\partial t} \right) \left(g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s} \right) = 2\gamma^{\delta\epsilon} \left(g_{ij} \frac{\partial^2 f^i}{\partial x^\delta \partial s} \frac{\partial^2 f^j}{\partial x^\epsilon \partial s} - \frac{1}{2} R_{ijkl} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial x^\delta} \frac{\partial f^k}{\partial s} \frac{\partial f^l}{\partial x^\epsilon} \right) \quad (25)$$

where R_{ijkl} is the curvature tensor of the target manifold N . (A more detailed computation will be given in the next step.) Since we assume that the latter has nonpositive sectional curvature, we conclude

$$\left(\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} - \frac{\partial}{\partial t} \right) \left(g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s} \right) \geq 0. \quad (26)$$

One such family of solutions is obtained by a time shift,

$$f(x, t, s) := f(x, t + s). \quad (27)$$

We use this to obtain

Lemma 4.1

$$\sup_{x \in M} g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \quad (28)$$

is nonincreasing in t for a solution of (24).

Proof: Applying (26) to the family (27) yields

$$\left(\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} - \frac{\partial}{\partial t} \right) \left(g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \right) \geq 0, \quad (29)$$

and the maximum principle for subsolutions of parabolic equations then implies the result. \square

2. We consider

$$\eta(f) := \gamma^{\alpha\beta} g_{ij}(f(x, t)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \quad (30)$$

As in (26), we want to apply the operator $\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} \frac{\partial}{\partial t}$ to this expression. This time, however, we also have to deal with derivatives of the domain metric. In order to simplify the computation, we use the standard device of orthonormal frames at the point under consideration. For the target, we may assume $g_{ij} = \delta_{ij}$, $g_{ij,k} = 0$. For the domain, we may also assume $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$, but not necessarily also the vanishing of the first derivatives. We then compute, using subscripts for partial derivatives,

$$\begin{aligned} & \left(\frac{\partial^2}{\partial x^\delta \partial x^\delta} \frac{\partial}{\partial t} \right) \eta(f) \\ &= f_{x^\alpha x^\delta}^i f_{x^\alpha x^\delta}^i + \gamma^{\alpha\beta}{}_{,\delta} (f_{x^\alpha x^\delta}^i f_{x^\beta}^i + f_{x^\alpha}^i f_{x^\beta x^\delta}^i) + \gamma^{\alpha\beta}{}_{,\delta\delta} f_{x^\alpha}^i f_{x^\beta}^i \\ &- R_{ijkl} f_{x^\alpha}^i f_{x^\delta}^j f_{x^\alpha}^k f_{x^\delta}^l \end{aligned} \quad (31)$$

where we have again used the equation (24). Using the Schwarz inequality to handle the terms with first derivatives of the domain metric, the nonpositivity of the curvature of N and rewriting the result in general coordinates, we therefore obtain

$$\left(\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} \frac{\partial}{\partial t} \right) \eta(f) \geq -c\eta(f) + \frac{1}{2} |D^2 f(\cdot, t)|^2 \quad (32)$$

with some constant c . In particular, $\eta(f)$ satisfies a linear differential inequality, and we therefore obtain

$$\eta(f(x, t)) \leq c \sup_{t_0 \leq \tau \leq t} \int_M \eta(f(\cdot, \tau)), \quad (33)$$

for any $t_0 > 0$, see e.g. [9], Section 3.3.

3. Next, as in [12], using Jacobi field estimates (see e.g. [9], Section 2.5 and in particular formula (2.5.6) and the one preceding it), we obtain

$$\int_M \eta(f(., t)) \leq c \int_M (\tilde{d}^2(f(., t), f^0) - \inf_{z \in M} \tilde{d}^2(f(z, t), f^0(z))) + c \quad (34)$$

where $\tilde{d}(f(., t), f^0(.))$ is the homotopy distance between the initial map $f^0 = f(., 0)$ and the map $f(., t)$ at time t ; the homotopy distance $\tilde{d}(f(x, t), f^0(x))$ for these two homotopic maps is given by the length of the shortest geodesic from $f(x, t)$ to $f^0(x)$ in the homotopy class of curves determined by the homotopy between the maps.

Also, these Jacobi field estimates yield

$$\gamma^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \tilde{d}^2(f(., t), f^0) \geq -c \tilde{d}(f(., t), f^0). \quad (35)$$

4. We can now complete the first step and prove long time existence of a solution of (24).

Since in (33), we take a supremum over different times τ , we have to control the behavior of our solution at different times against each other. We have by the triangle inequality

$$\tilde{d}^2(f(., \tau), f^0) \leq 2\tilde{d}^2(f(., t), f(., \tau)) + 2\tilde{d}^2(f(., t), f^0). \quad (36)$$

Also,

$$\tilde{d}(f(., t), f(., \tau)) \leq |t - \tau| \sup_{\tau \leq \sigma \leq t} |f_t(., \sigma)| \leq c|t - \tau| \quad (37)$$

where the last inequality follows from Lemma 4.1. With these inequalities at hand, we can use (33) and (34) to obtain for the norm of the first derivative df w.r.t. the spatial variable x

$$|df(x, t)| \leq c \left(\int_M \tilde{d}^2(f(., \tau), f^0) \right)^{1/2} + c \quad (38)$$

and from this then also

$$|df(x, t)| \leq c \sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)) + c. \quad (39)$$

Using (37) then yields

$$|df(x, t)| \leq c(1 + t). \quad (40)$$

(40) and Lemma 4.1 yield C^1 -bounds for our solution $f(x, t)$ of (24). We thus look at (17) as an inhomogeneous linear parabolic system with bounded right hand side. We can then apply the regularity theory for solutions of linear parabolic equations to get $C^{2,\alpha}$ -bounds by the standard bootstrapping argument. Such bounds then imply closedness of the interval of existence, hence global existence. Thus, we have shown

Lemma 4.2 *For a target manifold N of nonpositive sectional curvature, the solution $f(x, t)$ of (24) exists for all $t \geq 0$.*

We now turn to the second step of the proof, the convergence of the solution $f(x, t)$ of (24) to an affine harmonic map for $t \rightarrow \infty$. Here, we need to use the assumption of topological nontriviality as expressed in our theorem in addition to nonpositive target curvature (for necessary background material on nonpositive curvature, we may refer to, e.g., [10]). Again, we divide the reasoning into several substeps.

1. Let $x_0 \in M$ be a point where $\tilde{d}(f(y, \tau), f^0(y))$ attains its minimum. Using (35) and applying the maximum principle on both the ball $B(x_0, R)$ of radius R about x_0 and on its complement $M \setminus B(x_0, R)$, we obtain

$$\sup_{y \in M} \tilde{d}^2(f(y, \tau), f^0(y)) \leq \sup_{z \in \partial B(x_0, R)} \tilde{d}^2(f(z, \tau), f^0(z)) + c(R) \sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)) \quad (41)$$

where the constant $c(R)$ depends on the radius R . The boundary term can be controlled as follows

$$\begin{aligned} & \sup_{z \in \partial B(x_0, R)} \tilde{d}^2(f(z, \tau), f^0(z)) \\ & \leq \tilde{d}^2(f(x_0, \tau), f^0(x_0)) + 2R \sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)) (|df(y, \tau)| + |df^0(y)|) \end{aligned} \quad (42)$$

Using (39), (41) and (42), we obtain for a suitable choice of $R > 0$

$$\sup_{y \in M} \tilde{d}^2(f(y, \tau), f^0(y)) \leq \inf_{y \in M} \tilde{d}^2(f(y, \tau), f^0(y)) + c \sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)). \quad (43)$$

2. Combining (34) and (43),

$$\int_M \eta(f(\cdot, t)) \leq c \sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)) + c. \quad (44)$$

Using then (33) gives the pointwise estimate

$$|df(x, t)| \leq c(\sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)))^{1/2} + c. \quad (45)$$

Therefore, for any $x_1, x_2 \in M$, letting \tilde{f} denote the lift to universal covers,

$$d(\tilde{f}(x_1, t), \tilde{f}(x_2, t)) \leq c(\sup_{y \in M} \tilde{d}(f(y, \tau), f^0(y)))^{1/2} + c. \quad (46)$$

3. The essential point of the proof will be to exclude that for some sequence $t_n \rightarrow \infty$ and for some, and by (43) then for all, $y \in M$,

$$\tilde{d}(f(y, t_n), f^0(y)) \rightarrow \infty. \quad (47)$$

For $x \in M$, we let γ_x^n be the geodesic from $f^0(x)$ to $f(x, t_n)$ in the right homotopy class, i.e., the one determined by the homotopy between the maps f^0 and $f(\cdot, t_n)$. Their length T_n will then go to infinity, if (47). In fact, while the length depends on i , by (43), this is inessential.

Since N has nonpositive sectional curvature, the distance

$$d(\gamma_{x_1}^n(\tau), \gamma_{x_2}^n(\tau)) \quad (48)$$

is a convex function of τ . Since by (46), this distance grows at most like $(T_n)^{1/2}$, it must be bounded. Therefore, the geodesic rays γ_{x_i} that are the limits of $\gamma_{x_i}^n$ for $n \rightarrow \infty$ (perhaps after a selection of a subsequence) satisfy

$$\kappa(x_1, x_2, \tau) := d(\gamma_{x_1}(\tau), \gamma_{x_2}(\tau)) \leq d(\gamma_{x_1}(0), \gamma_{x_2}(0)) \quad (49)$$

for all positive τ .

There are then two possibilities: Either κ is decreasing in t or constant. In fact, we may always assume the latter, by the following observation. Since (49) holds for any two points x_1, x_2 , we then also conclude that

$$\eta(f(x, t)) \quad (50)$$

is a nonincreasing function of t for every x , and it has to decrease for some x unless κ is constant in t for any two points. When, however, we choose our initial values f^0 as a harmonic map, i.e., one that minimizes $\int_M \eta(f(\cdot))$, then $\eta(f(x, t))$ can only be a nonincreasing function of t for each x if it is constant.

Now, when $\kappa(x_1, x_2, \tau)$ is a constant function of τ , it generates a flat strip, since N has nonpositive sectional curvature.

4. Also, if f^0 is energy minimizing, then for any $t \geq 0$, the map $f^t(x) := \gamma_x(t)$ is also energy minimizing, by the same reasoning. We shall now use these energy minimizing maps to track our sequence $f(\cdot, t)$ and to get time independent estimates. We take a sequence $t_n \rightarrow \infty$ as above and write f^n in place of f^{t_n} . From the preceding constructions we obtain, in case (479) holds,

$$\tilde{d}(f(\cdot, t_n), f^n) \leq c(\tilde{d}(f(\cdot, t_n), f^0))^{1/2} + c. \quad (51)$$

We wish to get rid of the first term on the right hand side, i.e., we want f^n to track $f(\cdot, t_n)$ uniformly. That will then give us some control on the first derivatives of those maps w.r.t. x .

We can repeat the construction with f^n in place of f^0 . We have two possibilities. Either after finitely many steps, we find some energy minimizing map \hat{f}^n with

$$\tilde{d}(f(\cdot, t_n), \hat{f}^n) \leq c, \quad (52)$$

or we generate a new flat direction from strips between geodesics rays of constant distance as above in each step. In that case, however, after finitely many steps, we have exhausted all possible directions, and N must be flat. In that case, it is elementary to track $f(\cdot, t_n)$ also in the desired manner, and in fact, we are then dealing with linear parabolic equations which is much easier than the nonlinear case. Thus, in either case, we may assume (52).

We may then apply the reasoning leading to (39) with the variable map \hat{f}^n in place of f^0 to obtain

$$|df(x, t)| \leq c \sup_{y \in M} \tilde{d}(f(y, \tau), \hat{f}^n(y)) + c \leq c. \quad (53)$$

5. With Lemma 4.1 and (53), we have uniform estimates for all first derivatives of $f(x, t)$, i.e., estimates that do not depend on t . Linear elliptic parabolic regularity theory then also yields higher order estimates, and we can then find a sequence $t_n \rightarrow \infty$ for which $f(\cdot, t_n)$ converges smoothly to some smooth map f_∞ in the right homotopy class. It remains to show that f_∞ is affine harmonic.
6. We recall (25) for the family $f(x, t, s) := f(x, t + s)$, that is,

$$\begin{aligned} & \left(\gamma^{\delta\epsilon} \frac{\partial^2}{\partial x^\delta \partial x^\epsilon} - \frac{\partial}{\partial t} \right) \left(g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \right) \\ &= 2\gamma^{\delta\epsilon} \left(g_{ij} \frac{\partial^2 f^i}{\partial x^\delta \partial t} \frac{\partial^2 f^j}{\partial x^\epsilon \partial t} - \frac{1}{2} R_{ijkl} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial x^\delta} \frac{\partial f^k}{\partial t} \frac{\partial f^l}{\partial x^\epsilon} \right). \end{aligned} \quad (54)$$

Since we know from Lemma 4.1 that $g_{ij} \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t}$ stays bounded in t , and since both terms on the right hand side of (54) are nonnegative, they both have to converge to 0 for $t \rightarrow \infty$. The asymptotic vanishing of the first term means that

$$\frac{\partial f(x, t)}{\partial t} \quad (55)$$

converges to a parallel section $v(x)$ along f_∞ for $t \rightarrow \infty$. This, however, is excluded in the assumptions of our theorem. Therefore,

$$\frac{\partial f(x, t)}{\partial t} \rightarrow 0 \text{ for } t \rightarrow \infty. \quad (56)$$

Thus, in the limit $t \rightarrow \infty$, the temporal derivative disappears in (24), and the elliptic system that we want to solve remains. This, together with the smooth convergence of $f(\cdot, t_n)$ to f_∞ , shows that f_∞ solves the elliptic system, i.e., it is affine harmonic. This completes the proof of our main theorem. In fact, it is not hard to show now that the solution $f(\cdot, t)$ of the parabolic system converges to the solution f_∞ of the elliptic system as $t \rightarrow \infty$.

Remarks:

1. Naturally, one can also treat the Dirichlet problem for affine harmonic maps. Here, one could either use the method of [12] or the general approach developed by von Wahl [15, 16] for parabolic systems that does not need a variational structure. When Dirichlet boundary values are given, they prevent a solution from eternally moving around the target manifold. Thus, the main problem that we had to overcome in the proof of our main theorem and for which we needed an additional topological assumption besides the geometric condition of nonpositive curvature is not present in the Dirichlet boundary value problem. Of course, boundary regularity then is an issue that needs treatment, but this can be achieved by the methods of the aforementioned papers.
2. It should be possible and of interest in affine geometry to extend the method of Grunau and Kühnel [8] to show the existence of affine harmonic maps from a complete affine to a complete Riemannian manifold.

Acknowledgements The second author is grateful to J. Jost for posing the problem and for stimulating discussions. The second author was supported by the The Scientific and Technologic Research Council of Turkey, 2219 Fellowship and Max Planck Institute of Mathematics in the Sciences.

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